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On the Bianchi–Bäcklund construction for affine minimal surfaces

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Abstract. New affinely minimal surfaces are constructed with the use of Bäcklund's theorem. Corresponding affine Bäcklund transformation is studied in some detail.

1. Introduction

In the second half of the nineteenth century Bianchi and Bäcklund developed the technique of generating new surfaces of constant negative curvature in E^3 by the solution of a completely integrable system of first-order partial differential equations. Because there is an 'almost' one-to-one correspondence between solutions of the sine-Gordon equation $x_{,uv} = \sin x$ and the (local) surfaces of constant Gaussian curvature [1], the Bianchi-Bäcklund construction furnishes a way to generate new solutions of the sine-Gordon equation from a given one. This method, known as the Bäcklund transformation, has been recently used in the study of many non-linear evolution equations [2].

Chern and Terng's affine analogue of classical Bäcklund theorem [3] allows us to follow the Bianchi-Bäcklund idea in the case of affine minimal surfaces in the real affine space A^3. In the first part of this paper we present the analytic form of affine Bäcklund transformation (ABT) and discuss some of its properties. Then we give a few examples of constructing new families of solutions of the system (1) as well as new affine minimal surfaces.

Equations (1), being the integrability conditions of some linear problem and possessing a Bäcklund transformation [4], seems to be a candidate for complete solvability. But the linear problem as well as Bäcklund transformation is parameter independent (parameters appearing in [4] can be removed by gauge transformation). Lately D Levi (private communication) showed that Lie-Bäcklund symmetries of the system (1) are inessential generalisations of the transformation (11) and (12) and cannot be used to produce a parameter dependent linear problem. Thus the question of complete solvability of (1) remains open.

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2. Affine Bäcklund transformation

The local and implicit description of the so-called hyperbolic affine surfaces in A^3 is given in terms of the affine metric $II = F\,\mathrm{d}u\,\mathrm{d}v$ and the Fubini-Pick form $P = \frac{1}{2}A(\mathrm{d}u)^3 + \frac{1}{2}B(\mathrm{d}v)^3$, where $F > 0$, A, B are some real functions of the asymptotic coordinates u and v [5]. F, A and B obey the affine analogues of the Gauss-Mainardi-Codazzi equations. The condition of affine minimality $FF_{,uv} - F_{,u}F_{,v} + AB = 0$ and the GMC equations lead to the system of three non-linear partial differential equations [4]

$$\begin{aligned} &FF_{,uv} - F_{,u}F_{,v} + AB = 0 \\ &AB_{,u} - F_{,v}A_{,v} + FA_{,vv} = 0 \\ &A_{,v}B - F_{,u}B_{,u} + FB_{,uu} = 0 \end{aligned} \tag{1}$$

(the comma denotes differentiation). From the existence and uniqueness theorem on affine surfaces [5], $F > 0$, A and B satisfying (1) define the local affine minimal surface up to the unimodular affine transformation.

Let $r(u, v)$ be hyperbolic affine minimal surface parametrised by its asymptotic curves and let F, A and B be affine invariants of this surface. We may formulate an affine analogue of Bäcklund's theorem [3] as follows.

If $s(u, v) > 0$, $p(u, v)$ and $q(u, v)$ satisfy the system of first-order partial differential equations

$$\begin{aligned} &p_{,u} = -1 - s - p(\ln F)_{,u} \qquad && p_{,v} = q(B/F) \\ &q_{,u} = p(A/F) && q_{,v} = -1 + s - q(\ln F)_{,v} \\ &s_{,u} = p(A_{,v}/F) && s_{,v} = -q(B_{,u}/F) \end{aligned} \tag{2}$$

then the surface

$$r^*(u, v) = r(u, v) + p(u, v)r_{,u}(u, v) + q(u, v)r_{,v}(u, v) \tag{3}$$

is affine minimal too. (u, v) are asymptotic coordinates on $r^*(u, v)$.

Affine invariants of the new surface $r^*(u, v)$ are

$$F^* = sF \qquad A^* = -sA + qA_{,v} \qquad B^* = -sB - pB_{,u}. \tag{4}$$

System (2) is completely integrable in the Frobenius sense.

All the above statements can be checked directly without referring to [3].

The ABT on the level of the surfaces gives us the method of constructing new affine minimal surfaces from the known ones. One only needs to solve the system (2) and apply the formula (3). However, we have a Bäcklund transformation (BT) on the level of invariants of the surface (or non-linear equations) too. It is a perfect example of the Bäcklund transformation in the sense of soliton theory: s, p, q are the so-called pseudopotentials, they satisfy an overdetermined system (2), the compatibility conditions of which are equivalent to the original system (1). Given a solution of (1) we can put it into (2), solve for s, p, q and by applying (4) we can construct a solution of (1). Hence, on picking out initial values $s(u_0, v_0) = s_0$, $p(u_0, v_0) = p_0$, $q(u_0, v_0) = q_0$ we define a unique local solution of (2) and that is why the formula

$$(F, A, B) \longmapsto BT(F, A, B) = (F^*, A^*, B^*) \tag{5}$$

gives us a three-parameter family of solutions of (1). (The fourth parameter k appearing in [4] turns out not to be a true one.) Because of the one-to-one correspondence between solutions of (1) (with $F>0$) and affine minimal surfaces (up to unimodular affine transformation) in order to obtain some information about ABT we may investigate the BT (5) on the level of invariants of the surface as well. Bearing in mind these geometric applications, we will consider a proper BT (i.e. with $s>0$) only.

Let us start with two simple observations.

(i) If (F^*, A^*, B^*) is an image of (F, A, B) under BT with pseudopotentials s, p, q then (F, A, B) is an image of (F^*, A^*, B^*) under BT with pseudopotentials $s^*=1/s$, $p^*=p/s$, $q^*=-q/s$. That means that s^*, p^*, q^* give the inverse BT:

$$(F, A, B) \xmapsto{s,p,q} (F^*, A^*, B^*) \xmapsto{s^*,p^*,q^*} (F, A, B).$$

(ii) The following diagram,

$$\begin{tikzcd} & (\hat{F}, \hat{A}, \hat{B}) \arrow[dr, "s^*{,}\,p^*{,}\,q^*"] & \\ (F, A, B) \arrow[ur, "\hat{s}{,}\,\hat{p}{,}\,\hat{q}"] \arrow[dr, "\bar{s}{,}\,\bar{p}{,}\,\bar{q}"'] & & (F^*, A^*, B^*) \\ & (\bar{F}, \bar{A}, \bar{B}) \arrow[ur, dashed, "s{,}\,p{,}\,q\,=\,?"'] & \end{tikzcd}$$

where the full arrows represent BT with indicated pseudopotentials, can be closed. In point of fact, if we take

$$s=(s^*\hat{s})/\bar{s} \qquad p=p^*+s\bar{p}-s^*\hat{p} \qquad q=q^*-s\bar{q}+s^*\hat{q}$$

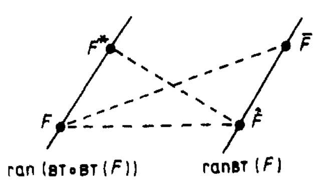
long but straightforward calculations show that they define BT

$$(\bar{F}, \bar{A}, \bar{B}) \xmapsto{s,p,q} (F^*, A^*, B^*).$$

This feature of permutability of BT (5) allows us to prove the theorem on the range of consecutive applications of BT. Let us denote by $\mathrm{ranBT}(F, A, B)$ the range of BT applied to (F, A, B), i.e. the set of all (F^*, A^*, B^*) which can be obtained from (F, A, B) via BT with all possible s, p, q.

Theorem. $\mathrm{ran}(\mathrm{BT}\circ\mathrm{BT}\circ\mathrm{BT}(F, A, B)) = \mathrm{ranBT}(F, A, B)$.

Proof. Let us start with some F satisfying (1) (we shall omit A and B for convenience), then construct $\mathrm{BT}(F)$. The permutability of (5) (see (ii) above) assures us that $\mathrm{ranBT}(\hat{F})=\mathrm{ranBT}(\bar{F})$ for every pair $\hat{F}, \bar{F}\in\mathrm{ranBT}(F)$. Because $F\in\mathrm{ran}(\mathrm{BT}\circ\mathrm{BT}(F))$ we obtain the following picture



Now permutability of (5) applied to $\hat{F}$ gives us $\mathrm{ranBT}(F^*)=\mathrm{ranBT}(F)$ for every $F^*\in\mathrm{ranBT}(\hat{F})=\mathrm{ran}(\mathrm{BT}\circ\mathrm{BT}(F))$. So we obtain the desired result.

The theorem and its proof show us that given a solution F, A, B of (1) one can construct two three-parameter families of solution of (1): ranBT(F, A, B) and ran(BT $\circ$ BT(F, A, B). Consecutive applications of BT (5) do not give us new solutions: BT transforms ranBT(F, A, B) onto ran(BT $\circ$ BT(F, A, B)) and conversely. Thus we can see that, as far as the range of ABT is concerned, it is drastically different from the classical Bäcklund transformation for pseudospheres. The latter possesses some external parameter (it is a parameter dependent Bäcklund transformation) and its consecutive applications allow us to construct more and more new pseudospherical surfaces (or solutions of the sine-Gordon equation).

Let us make some observations concerning the mutual position of ranBT(F, A, B) and ran(BT $\circ$ BT(F, A, B)). Though they can intersect in general, in a generic position ranBT(F) $\cap$ ran(BT $\circ$ BT(F)) $= \varnothing$. This results from

(i) $(F, A, B) \in$ ranBT(F) $\cap$ ran(BT $\circ$ BT(F)) $\Rightarrow A = 0$

(ii) $(\bar{F}, \bar{A}, \bar{B}) \in$ ranBT(F, $A = 0$, B) $\Rightarrow \bar{A} = 0$.

Now we present another formulation of BT (5) which is sometimes more convenient than the original one. When $A_{,v}B_{,u} \neq 0$ formulae (2) and (4) are equivalent to the system of first-order linear partial differential equations for unknowns F^*, A^* and B^*:

$$\begin{aligned}
&\text{(a)} && A^*_{,v} + A_{,v} = 0 \\
&\text{(b)} && A^*_{,u}A_{,v}F - A^*A_{,uv}F + (A_{,u}A_{,v} - AA_{,uv})F^* = 0 \\
&\text{(c)} && B^*_{,u} - B_{,u} = 0 \\
&\text{(d)} && B^*_{,v}B_{,u}F - B^*B_{,uv}F + (B_{,u}B_{,v} - BB_{,uv})F^* = 0 \\
&\text{(e)} && F^*_{,u}B_{,u} - F^*B_{,uu} + B^*A_{,v} = 0 \\
&\text{(f)} && F^*_{,v}A_{,v} - F^*A_{,vv} + A^*B_{,u} = 0.
\end{aligned} \tag{6}$$

Pseudopotentials s, p, q are given by

$$s = \frac{F^*}{F} \qquad p = -\frac{FB^* + F^*B}{FB_{,u}} \qquad q = \frac{FA^* + F^*A}{FA_{,v}}. \tag{7}$$

When $A_{,u}A_{,v} - AA_{,uv} \neq 0$ and $B_{,u}B_{,v} - BB_{,uv} \neq 0$ we can solve some equations from the system (6) and reduce construction of BT (5) to the process of solution of one 'partial-ordinary' differential equation.

Corollary. When $A_{,u}A_{,v} - AA_{,uv} \neq 0$ and $B_{,u}B_{,v} - BB_{,uv} \neq 0$, system (6) is equivalent to the differential equation for two functions of one variable $g(u)$, $h(v)$

$$\begin{aligned}
&(hB_{,uv} - h'B_{,u})(A_{,u}A_{,v} - AA_{,uv}) - (gA_{,uv} - g'A_{,v}) \\
&\qquad \times (B_{,u}B_{,v} - BB_{,uv}) = 2(A_{,u}A_{,v} - AA_{,uv})(B_{,u}B_{,v} - BB_{,uv})
\end{aligned} \tag{8}$$

with the ansatz

$$A^*(u, v) = -A(u, v) + g(u) \qquad B^*(u, v) = B(u, v) + h(v)$$

$$F^* = F\left(1 - \frac{g'A_{,v} - gA_{,uv}}{A_{,u}A_{,v} - AA_{,uv}}\right).$$

In the above formulae $g' = \mathrm{d}g/\mathrm{d}u$, $h' = \mathrm{d}h/\mathrm{d}v$.

Proof. From (6a) and (6c) we obtain $A^*(u, v) = -A(u, v) + g(u)$ and $B^*(u, v) = B(u, v) + h(v)$.

(6*b*) is equivalent to

$$F^* = F\left(1 - \frac{g' A_{,v} - g A_{,uv}}{A_{,u} A_{,v} - A A_{,uv}}\right). \tag{*}$$

Inserting this into (6*f*) we obtain the identity. Analogously (6*d*) is equivalent to

$$F^* = F\left(-1 - \frac{h' B_{,u} - h B_{,uv}}{B_{,u} B_{,v} - B B_{,uv}}\right) \tag{**}$$

which converts (6*e*) into the identity.

Equation (8) guarantees that (*) and (**) give the same F^*.

For applications it may be useful to know explicit formulae for two consecutive ABT. ABT applied to the surface $r^*(u, v)$ given by (3) leads to

$$\begin{aligned} \hat{r} &= r^* + p^* r^*_{,u} + q^* r^*_{,v} \\ &= r + (p - sp^*) r_{,u} + (q + sq^*) r_{,v} + (pq^* + qp^*) r_{,uv}. \end{aligned} \tag{9}$$

From (7) we obtain

$$\begin{aligned} p - sp^* &= (FB_{,u})^{-1}((\hat{F} - F)B^* + (\hat{B} - B)F^*) \\ q + sq^* &= (FA_{,v})^{-1}((F - \hat{F})A^* + (A - \hat{A})F^*) \\ pq^* + qp^* &= (FA_{,v}B_{,u})^{-1}(F(\hat{A}B^* - \hat{B}A^*) + F^*(\hat{A}B - \hat{B}A) + \hat{F}(A^*B - B^*A)). \end{aligned} \tag{10}$$

Before proceeding to examples of new affine minimal surfaces obtained through ABT let us briefly describe symmetries of the system (1). It is a system of homogeneous equations. Thus it has a one-parameter symmetry transformation

$$(F, A, B) \xmapsto{k} (\bar{F}, \bar{A}, \bar{B}) = (kF, kA, kB). \tag{11}$$

This transformation commutes with BT (5).

System (1) also possesses the two-parameter symmetry transformation

$$(F, A, B) \xmapsto{m,n} (\bar{F}, \bar{A}, \bar{B}) = \left[\frac{1}{mn} F\left(\frac{u}{m}, \frac{v}{n}\right), m^{-3} A\left(\frac{u}{m}, \frac{v}{n}\right), n^{-3} B\left(\frac{u}{m}, \frac{v}{n}\right)\right]. \tag{12}$$

This transformation commutes with BT provided we simultaneously transform pseudopotentials: $s \mapsto s$, $p \mapsto mp$, $q \mapsto nq$.

Let us sum up now. Given affine minimal surface $r(u, v)$ parametrised by asymptotic coordinates we can construct explicitly two three-parameter families of affine minimal surfaces provided we are able to solve the system (2), (6) or (8). Taking into account that symmetry transformation (11) corresponds to $r(u, v) \mapsto k^{2/3} r(u, v)$ and (12) is simply the change of variables (on the surface level) we can use (11) to get new surfaces too.

3. Examples

We present here a few examples of ABT in action. We construct two families ranBT(·), ran(BT ∘ BT(·)) of affine minimal surfaces starting with some classical minimal surfaces: Enneper surface and Thomsen surfaces as well as the 'travelling wave' surface found by ourselves. If F, A, B describe the original surface then we use F^*, A^*, B^* for surfaces belonging to $\mathrm{ran}\,\mathrm{BT}(F, A, B)$ and $\bar{F}, \bar{A}, \bar{B}$ for surfaces from $\mathrm{ran}(\mathrm{BT} \circ \mathrm{BT}(F, A, B))$. a, b, c, d, e, f are real constants.

I: $$AB = 0 \Rightarrow A^* B^* = 0.$$

These are affine minimal surfaces with vanishing affine curvature $\tilde{K} = 0$ (affinely flat).

II: $$A_{,v} B_{,u} = 0 \Rightarrow A^*_{,v} B^*_{,u} = 0.$$

These are affine minimal surfaces with vanishing total affine curvature $K = 0$ (total affinely flat).

III: $$A_{,v} = 0 = B_{,u} \Rightarrow A^*_{,v} = 0 = B^*_{,u}.$$

These are singular affine spheres. In this case BT satisfies $N^* = -(1/s)N$, $s = \text{constant}$, where $N = (r_{,uv}/F)$, $N^* = (r^*_{,uv}/F^*)$ are affine normals [5]. Conversely, if $N^* = -(1/s)N$, $s = \text{constant} \neq 1$, then both $r(u, v)$ and $r^*(u, v)$ are singular affine spheres.

IV: $$A_{,u} = 0 = B_{,v} \text{ and } A_{,v} B_{,u} \neq 0 \Rightarrow A^*_{,u} = 0 = B^*_{,v}.$$

This is the case when BT in the form (6) is very efficient.

3.1. *Generalised Enneper surfaces*

The Enneper surface is given (in asymptotic coordinates) by [5]

$$\begin{aligned} x &= 3v - 3u^2 v + v^3 \\ y &= 3u - 3uv^2 + u^3 \\ z &= -6uv. \end{aligned} \tag{13}$$

Its affine invariants are

$$F = k(1 + u^2 + v^2) \qquad A = 2kv \qquad B = 2ku \tag{14}$$

with $k = 3\sqrt{6}$.

BT applied to (14) gives

$$\begin{aligned} F^* &= k[-\tfrac{1}{4}(2u + b)^2 + \tfrac{1}{4}(-2v + a)^2 + c] \\ A^* &= k(-2v + a) \qquad B^* = k(2u + b). \end{aligned} \tag{15}$$

BT applied to (15) gives

$$\begin{aligned} \bar{F} &= k[\tfrac{1}{4}(2u + e)^2 + \tfrac{1}{4}(2v + d)^2 + f] \\ \bar{A} &= k(2v + d) \qquad \bar{B} = k(2u + e). \end{aligned} \tag{16}$$

Applying BT once more we obtain (15) again. It is the illustration of the theorem. Substituting $e = d = 0, f = 1$ into (16) one obtains (14). Our general scheme allows us to construct explicitly every surface belonging to the families (15) and (16) for we know one of them—the Enneper surface (13).

3.2. Generalised Thomsen surfaces [5]

Minimal Thomsen surfaces are given (in asymptotic coordinates) by

$$\begin{aligned} x &= -(1-t^2)^{-1/2}(tv+\cos u \sinh v) \\ y &= (1-t^2)^{-1/2}(u+t\sin u\cosh v) \\ z &= \sin u \sinh v. \end{aligned} \tag{17}$$

Corresponding affine invariants are

$$\begin{aligned} &F = (1-t^2)^{-1/2}(\cosh v + t\cos u) \\ &A = (1-t^2)^{-1/2}\sinh v \qquad B = -t(1-t^2)^{-1/2}\sin u \end{aligned} \tag{18}$$

where t is a real constant.

BT applied to (18) gives

$$\begin{aligned} &F^* = (1-t^2)^{-1/2}[(c\cos u - b\sin u - 1)\cosh v + t(1-a\sinh v)\cos u] \\ &A^* = -(1-t^2)^{-1/2}(\sinh v + a) \qquad B^* = -t(1-t^2)^{-1/2}(\sin u + b). \end{aligned} \tag{19}$$

BT applied to (19) leads to

$$\begin{aligned} &\bar{F} = [(f\cos u + e\sin u + 1)\cosh v + t(1-d\sinh v)\cos u](1-t^2)^{-1/2} \\ &\bar{A} = (1-t^2)^{-1/2}(\sinh v + d) \qquad \bar{B} = -t(1-t^2)^{-1/2}(\sin u + e). \end{aligned} \tag{20}$$

Substituting $d=e=f=0$ into (20) we obtain minimal Thomsen surfaces (18).

Again, we are able to construct explicitly every surface belonging to the families (19) and (20).

V: $\qquad A_{,u}A_{,v} - AA_{,uv} \neq 0 \qquad$ and $\qquad B_{,u}B_{,v} - BB_{,uv} \neq 0.$

In this case the representation of BT given in the corollary (formula (8)) is most efficient. As an example we shall consider a travelling wave solution of (1)

$$F = k(u+v)^4 \qquad A = 2k(u+v)^3 \qquad B = 2k(u+v)^3. \tag{21}$$

BT applied to (21) gives

$$\begin{aligned} F^* &= k(u+v)[-u^3 - 3u^2v + 3uv^2 + v^3 - auv + \tfrac{1}{2}b(u-v) + c] \\ A^* &= -2k(u+v)^3 + k(4u^3 + au^2 + bu + c) \\ B^* &= 2k(u+v)^3 + k(-4v^3 + av^2 - bv + c). \end{aligned} \tag{22}$$

BT applied to (22) leads to

$$\begin{aligned} \bar{F} &= k(u+v)[(u+v)^3 + duv - \tfrac{1}{2}e(u-v) - f] \\ \bar{A} &= 2k(u+v)^3 + k(du^2 + eu + f) \\ \bar{B} &= 2k(u+v)^3 + k(dv^2 - ev + f). \end{aligned} \tag{23}$$

BT applied to (23) gives, according to the theorem, the family (22) again. Substituting $d=e=f=0$ into (23) we recover the travelling wave solution (21).

We can construct explicitly every surface belonging to families (22) and (23) because we know one special example of such a surface:

$$x = 5(u+v)^3 \qquad y = 3(u+v)^3(u-v) \qquad z = 2(u+v)^3(u^2 - 3uv + v^2)$$

corresponding to $k=30$ in (21).

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References

[1] Sym A 1985 *Geometric Aspects of the Einstein Equations and Integrable Systems* (*Lecture Notes in Phys.* **239**) ed R Martini (Berlin: Springer)

[2] Miura R M (ed) 1976 *Bäcklund Transformations* (*Lecture Notes in Math.* **515**) (Berlin: Springer)

[3] Chern S S and Terng C L 1980 *Rocky Mountain J. Math.* **10** 105–24

[4] Antonowicz M and Sym A 1985 *Phys. Lett.* **112A** 1–2

[5] Schirokow P A and Schirokow A P 1962 *Affine Differentialgeometrie* (Leipzig) (translated from Russian)